- Jacobian matrix of $f: \mathbb{R}^n \to \mathbb{R}^m$, $Df = \begin{pmatrix} -\nabla f_1- \\ \vdots \\ -\nabla f_m- \end{pmatrix}$
- differentiability of vector-valued multi-variable functions
  $\Leftrightarrow$ Each f_i are differentiable.
- Chain rule $\mathbb{R}^k \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^m$, $D(g\circ f)(a) = Dg(f(a)) \cdot Df(a)$

(idea of proof of chain rule)

Suppose $f: \Omega_1 (\subseteq \mathbb{R}^k) \to \mathbb{R}^n$ differentiable at $\vec{a}$.

$g: \Omega_2 (\subseteq \mathbb{R}^n) \to \mathbb{R}^m$ " " $\vec{b} = f(\vec{a})$.

Ω_2

For $x \in \Omega_1$, $f(x) - f(a) = Df(a)(x-a) + \mathcal{E}_f(x)$ (i)

$y \in \Omega_2$, $g(y) - g(b) = Dg(b)(y-b) + \mathcal{E}_g(y)$ (ii)

Put $y = f(x)$, $b = f(a)$, (i) to (ii)

$$g(f(x)) - g(f(a)) = Dg(f(a))(Df(a)(x-a) + \mathcal{E}_f(x)) + \mathcal{E}_g(f(x))$$

$$= \underbrace{Dg(f(a))\,Df(a)(x-a)}_{\text{linear in } x-a} + \underbrace{Dg(f(a))\,\mathcal{E}_f(x) + \mathcal{E}_g(f(x))}_{\text{Let it be } \mathcal{E}_{g\circ f}(x).}$$

We need to show that $\lim\limits_{x\to a} \dfrac{\|\mathcal{E}_{g\circ f}(x)\|}{\|x-a\|} = 0$

(roughly, since f is continuous, g is differentiable $\dfrac{\mathcal{E}_g(f(x))}{\|x-a\|} \to 0$ as $x \to a$

(also $\frac{\varepsilon_f(x)}{\|x-a\|} \to 0$ as $x \to a$ $\because$ f differentiable)

$\Rightarrow$ $g \circ f$ is differentiable, $D(g \circ f)(a) = Dg(f(a))\, Df(a)$.

Summary: Jacobian matrix.

① $f: \Omega (\subseteq \mathbb{R}) \to \mathbb{R}$ (one-variable. real-valued)

$Df(x) = \frac{df}{dx}$ (scalar. 1×1 matrix)

② $f: \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}$ (multi-variable, real valued)

$Df(x) = \nabla f(x)$ (gradient vector, $1 \times n$ matrix)

③ $f: \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^m$ (multi-variable, vector-valued)

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

$$Df(x) = \begin{pmatrix} -\nabla f_1- \\ \vdots \\ -\nabla f_m- \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$m \times n$ matrix

Chain rule in classical notations

$$(x_1 \cdots x_k) \xmapsto{f} (y_1 \cdots y_n) \xmapsto{g} (g_1 \cdots g_m)$$

$g_i = g_i(y_1 \cdots y_n)$ are functions on $y_1 \cdots y_n$

$y_j = f_j(x_1 \cdots x_k)$ " " $x_1 \cdots x_k$

If we regard g_i as functions on $x_1 \cdots x_k$

chain rule:

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_k} \end{pmatrix}$$

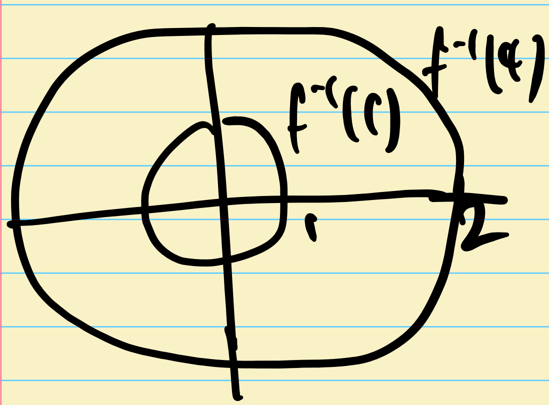
(i,j) entry:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_i}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial g_i}{\partial y_n}\frac{\partial y_n}{\partial x_j}.$$

Chain rule & level set.

<u>Recall</u> level set of $f: \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}$ at $c \in \mathbb{R}$ is $L_c = f^{-1}(c) = \{x \in \Omega \mid f(x) = c\}$.

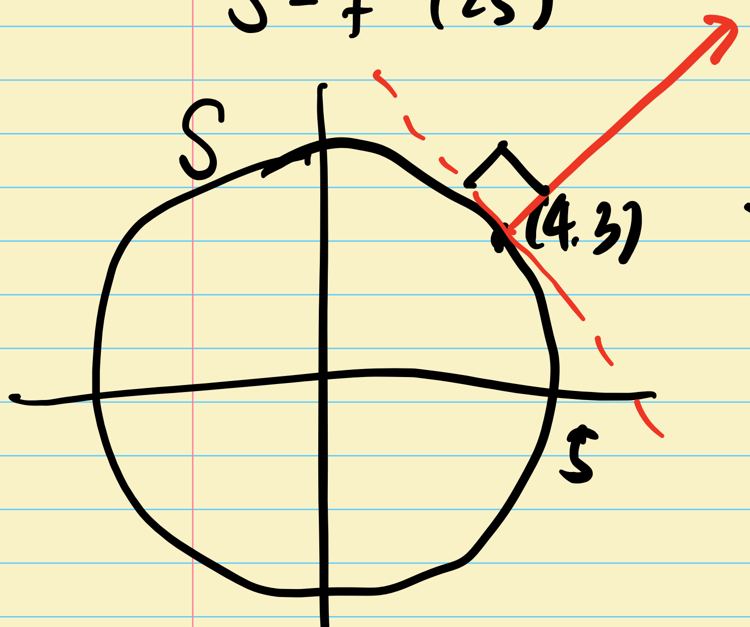
eg $f(x,y) = x^2 + y^2$



Thm Let $f : \Omega (\subseteq \mathbb{R}^n) \longrightarrow \mathbb{R}$, Ω is open.
Let $c \in \mathbb{R}$. $S = f^{-1}(c)$ and $a \in S$.
Suppose f is differentiable at a, $\nabla f(a) \neq 0$.
Then $\nabla f(a) \perp S$ at a.

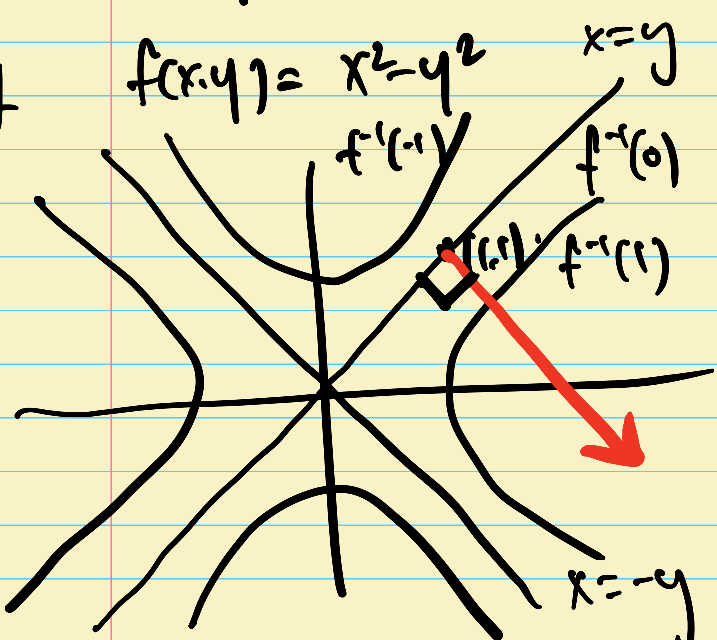
eg $f(x,y) = x^2+y^2$
$S = f^{-1}(25)$

$(4,3) \in S$.
$\nabla f = (2x, 2y)$
$\nabla f(4,3) = (8,6)$

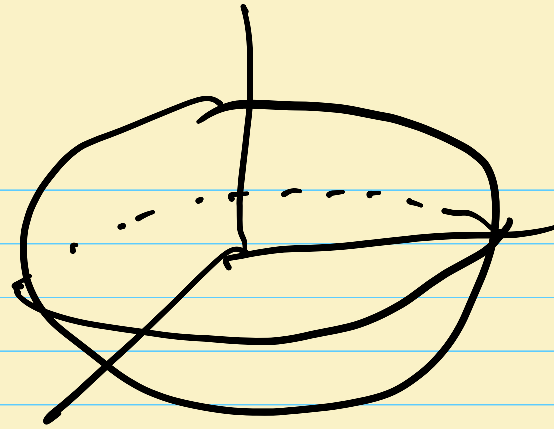


eg $f(x,y) = x^2-y^2$

$\nabla f = (2x, -2y)$
$\nabla f(1,1) = (2,-2)$



eg $S: x^2+4y^2+9z^2=22$



equation of tangent plane of S at $(3,1,1)$?

(sol) We need to find a normal vector.

Let $f(x,y,z) = x^2+4y^2+9z^2$.

$S = f^{-1}(22)$

By the theorem, $\nabla f(3,1,1)$ is $\perp$ S.

$\nabla f = (2x, 8y, 18z)$

$\nabla f(3,1,1) = (6, 8, 18)$

$\therefore$ An equation of tangent plane is

$$[(x,y,z)-(3,1,1)]\cdot(6,8,18)=0$$

$\Rightarrow$ $3x+4y+9z=22$. $\square$

(proof of theorem) $\gamma(t)$ be a curve on S s.t. $\gamma(0)=a$.

$\gamma(t)$ on $S=f^{-1}(c)$ $\Rightarrow$ $f(\gamma(t))=c$ a constant for all t.

By chain rule, $0 = \frac{d}{dt} f(\gamma(t)) = \nabla f(\gamma(t))\cdot\gamma'(t)$

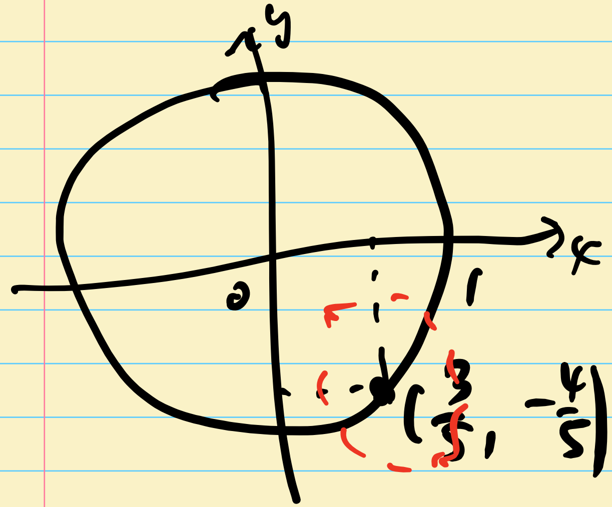
Put $t=0$, then $\nabla f(a)\cdot\gamma'(0)=0$.

$\therefore \nabla f(a) \perp$ any curve on S at a.

$\therefore \nabla f(a) \perp S$ at a. $\square$

Another application of chain rule : Implicity differentiation.

eg $C: x^2+y^2=1$. Try to find $\frac{dy}{dx}$ at $\left(\frac{3}{5}, -\frac{4}{5}\right)$



locally, y is a function on x.

<u>Method 1</u> Near $\left(\frac{3}{5}, -\frac{4}{5}\right)$,

$$y^2=1-x^2,\ y<0 \Rightarrow y=-\sqrt{1-x^2}$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(-\sqrt{1-x^2}\right)=-(-2x)\cdot\frac{1}{2\sqrt{1-x^2}}$$

$$=\frac{x}{\sqrt{1-x^2}}=\frac{3/5}{4/5}=3/4$$

Method 2 $x^2+y^2=1.$

Take $\frac{d}{dx}$ to both sides of

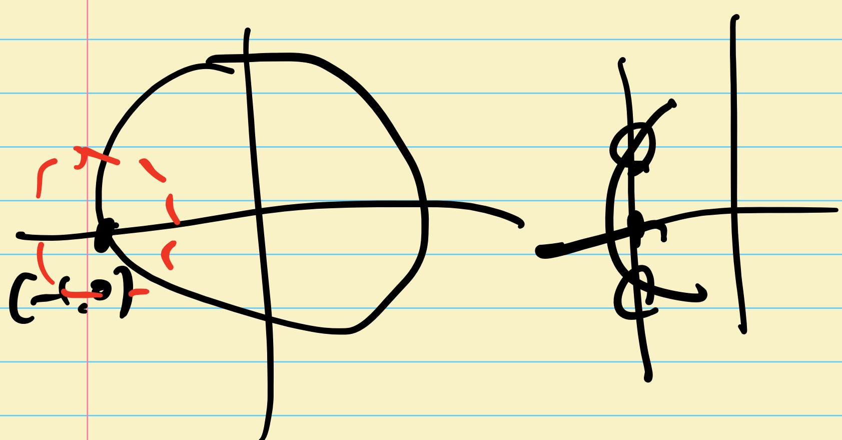
$$\Rightarrow 2x + \frac{dy^2}{dx} = 0$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

At $\left(\frac{3}{5}, -\frac{4}{5}\right)$, $2\left(\frac{3}{5}\right) + 2\cdot\left(-\frac{4}{5}\right)\cdot \left.\frac{dy}{dx}\right|_{\left(\frac{3}{5},-\frac{4}{5}\right)} = 0$

$$\therefore \left.\frac{dy}{dx}\right|_{\left(\frac{3}{5},-\frac{4}{5}\right)} = 3/4$$

Rmk Near $(-1,0)$



y is not a function on x

$\frac{dy}{dx}$ does make sense.

eg Consider $S: \underbrace{x^3+z^3+ye^{xz}+z\cos y}_{(*)}=0$

$(0,0,0)\in S$.

Given that z can be regarded as a function of variables x, y locally near $(0,0,0)$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0,0,0)$.

Rmk It is hard to express z in terms of x, y explicitly.

(sol) Take $\frac{\partial}{\partial x}$ to both sides of $(*)$,

$$3x^2+3z^2\cdot\frac{\partial z}{\partial x}+zy\cdot e^{xz}+\frac{\partial z}{\partial x}\cos y=0.$$

Put $(x,y,z)=(0,0,0)$

$$0+0+0+\left.\frac{\partial z}{\partial x}\right|_{(0,0,0)}\cdot 1=0$$

$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(0,0,0)}=0$$

Similarly, take $\frac{\partial}{\partial y}$ $\Rightarrow 0+3z^2\cdot\frac{\partial z}{\partial y}+e^{xz}+\frac{\partial z}{\partial y}\cos y$
$-z\sin y=0$

Put $(0,0,0)$ $\Rightarrow$ $0+0+1+\frac{\partial z}{\partial y}\cdot 1-0=0$

$\therefore \left.\frac{\partial z}{\partial y}\right|_{(0,0,0)} = -1.$

Rmk From the above computations,

$$\frac{\partial z}{\partial x} = -\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y},$$

$$\frac{\partial z}{\partial y} = \frac{z\sin y - e^{xz}}{2z+xye^{xz}+\cos y}.$$

whenever the denominators are nonzero.

Finding extrema (maximum & minimum)

Def Let $f: A(\subseteq \mathbb{R}^n) \to \mathbb{R}$, $a \in A$.

① f is said to have <u>global (absolute) maximum</u> at a if $f(a) \geq f(x)$ for $x \in A$.

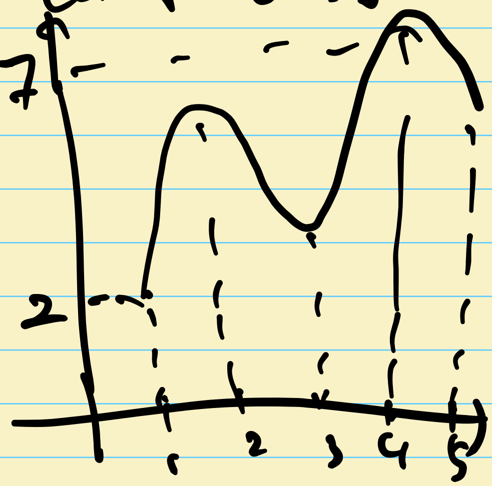
② f is said to have <u>local (relative) maximum</u> at a if $f(a) \geq f(x)$ for all $x \in A$ near a.

(i.e. $\exists \varepsilon > 0$ s.t. $f(a) \geq f(x)$ for all $x \in A \cap B_\varepsilon(a)$)

③ Similar definition for global minimum / local min.

Rmk global max/min $\Rightarrow$ local max/min.

eg Let $f : [1,5] \to \mathbb{R}$



Global max : at 4

Global min : at 1.

local max : at 2. 4

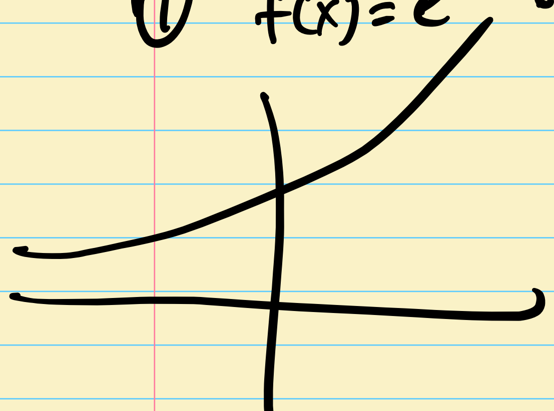
local min : 1. 3. 5.

max value = 7

min value = 2

Rmk Not every function has global max/min.

① $f(x) = e^x$ on $\mathbb{R}$.



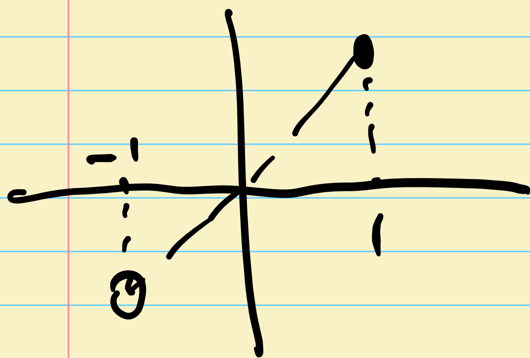
$\lim_{x \to \infty} f(x) = \infty$: no global maximum

$\lim_{x \to -\infty} f(x) = 0$ but $f(x) > 0$ $\forall x \in \mathbb{R}$

: no global minimum.

(Note that domain of f = $\mathbb{R}$)
not bounded

② $f(x) = x$ on $(-1, 1]$

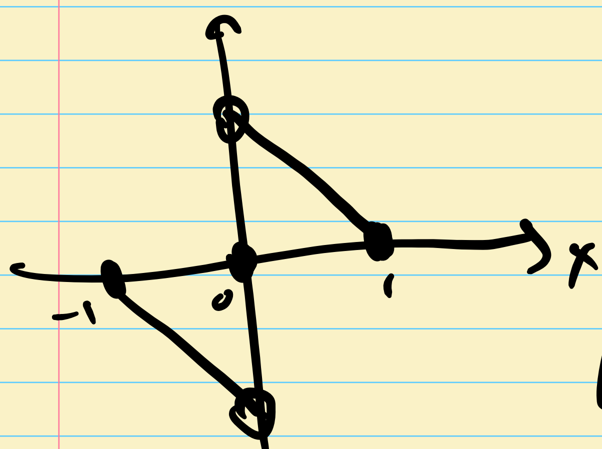


f has global max at 1, but no global min.

[Note that $A = (-1, 1]$ not closed]

③ $f : [-1, 1] \to \mathbb{R}$ define by

$$f(x) = \begin{cases} 1 - x & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \\ -1 - x & \text{if } x \in [-1, 0). \end{cases}$$



f has neither global max or min.

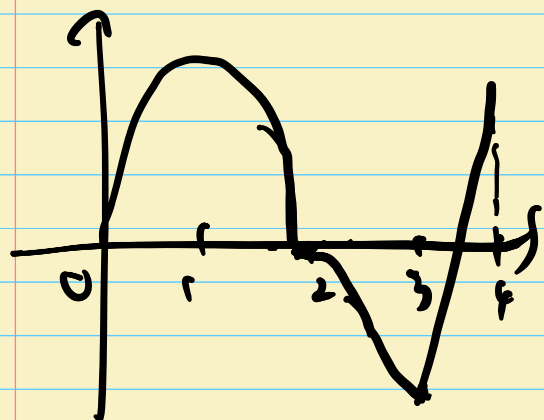
[Note that f is not continuous]

<u>Thm</u> (Extreme value theorem (EVT))

Let $A \subseteq \mathbb{R}^n$ be <u>closed and bounded</u> and $f : A \to \mathbb{R}$ is <u>continuous</u> function. Then f has global max and min.

Q How to locate max/min?

eg $f: [0,4] = A \longrightarrow \mathbb{R}$.



A is closed & bounded, f is continuous

$\overset{\text{Thm}}{\Longrightarrow}$ f has global max & min.

Recall one-variable calculus

extrema can only occur at

i) $f'(x)=0$: $x=1,2$ } critical points

ii) $f'(x)$ DNE : $x=3$ }

iii) $x \in \partial A$: $x=0,4$ } boundary points

Def $f: A \to \mathbb{R}$, $a \in \operatorname{int}(A)$
$(\subseteq \mathbb{R}^n)$

a is called a <u>critical point</u> of f if

① $\nabla f(a)$ DNE (i.e. $\frac{\partial f}{\partial x_i}(a)$ DNE for some i)

or ② $\nabla f(a) = 0$ (i.e. $\frac{\partial f}{\partial x_i}(a) = 0$ for all i)

Thm (First derivative test)

Suppose $f : A (\subseteq \mathbb{R}^n) \to \mathbb{R}$ has a local extremum at $a \in \mathrm{int}(A)$, then a is a critical point of f.

(pf) Suppose f has a local extremum at $a \in \mathrm{int}(A)$

If $\nabla f(a)$ DNE, nothing to prove.

If $\nabla f(a)$ exists, $\frac{\partial f}{\partial x_i}(a)$ exists,

For any $i = 1, \cdots, n$ let $g_i(t) = f(a + te_i)$

Note that $a \in \mathrm{int}(A)$

$\Rightarrow g_i(t)$ is defined near $t = 0$

we have

$$g_i'(0) = \frac{\partial f}{\partial x_i}(a)$$

f has local extremum at a

$\Rightarrow g_i$ " " " at $t = 0$

$\Rightarrow g_i'(0) = 0$ (one-variable calculus)

$\Rightarrow \frac{\partial f}{\partial x_i}(a) = 0$ $\quad \therefore \nabla f(\vec{a}) = 0.$

i.e. $\vec{a}$ is a critical point.

Strategy for finding extremum of $f: A (\subseteq \mathbb{R}^n) \to \mathbb{R}$

① Find critical points of f in $\text{int}(A)$

② Study f on boundary ∂A : Find max/min of f on ∂A.

③ Compare ① & ②

$\Rightarrow$ determine the extremum.